परमाणु ऊर्जा शिक्षण संस्था

(परमाणु ऊर्जा विभाग का स्वायत्त निकाय, भारत सरकार)

ATOMIC ENERGY EDUCATION SOCIETY

(An autonomous body under Department of Atomic Energy, Govt. of India)

CHAPTER -8

APPLICATIONS OF INTEGRATION

MODULE : 2/2

e -content

PREVIOUS KNOWLEDGE

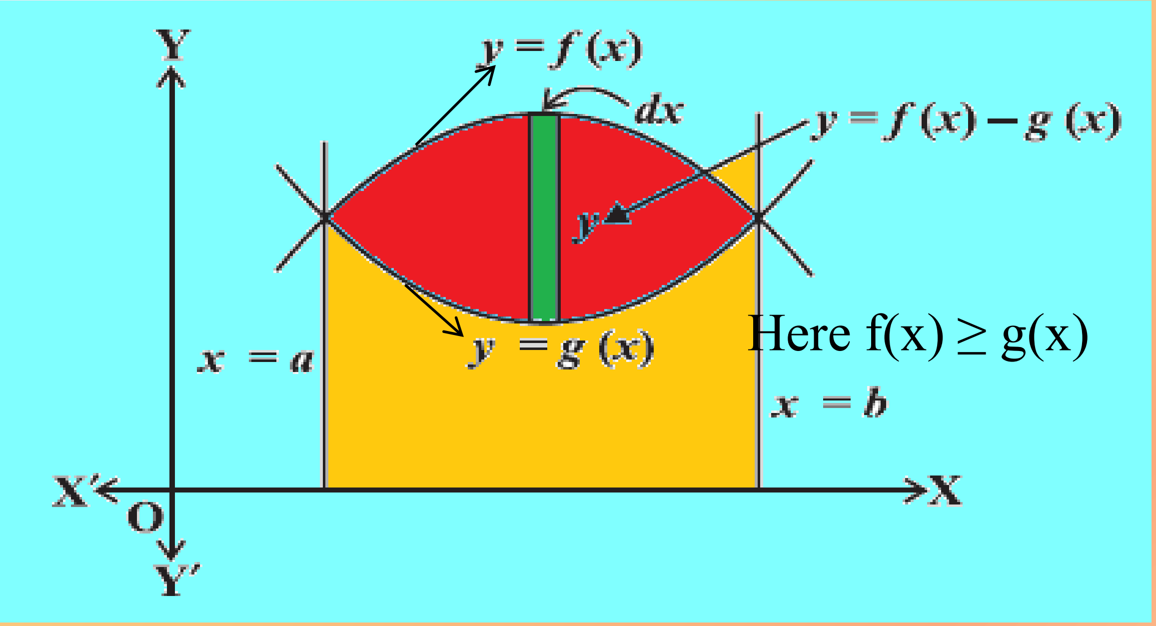
- Knowledge of finding the area bounded by a line and a curve.
- Standard equation of circle, parabola and ellipse .

AREA BETWEEN TWO CURVES

Consider two intersecting curves whose equations are y = f(x) and y = g(x) . Let us try to find the area bounded by these two curves. Let the point of intersection of these two curves be x = a and x = b .

Assume that f(x) ≥ g(x) . For finding the elementary area take an elementary strip with height f(x) –g(x) and width dx . Then if dA denote the elementary area dA = [f(x) –g(x)] dx. Hence total area

A = $\int_a^b [f(x) - g(x)]dx$



$$A = \int_a^b f(x)dx - \int_a^b g(x)dx$$

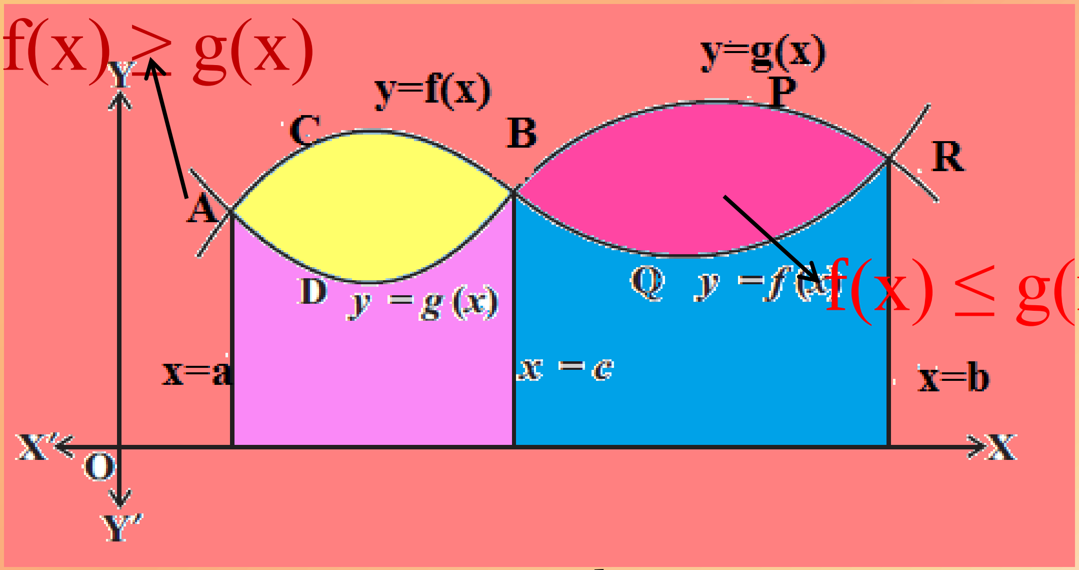
A = [area bounded by $y = f(x)$, x-axis and the lines $x = a$, $x = b$]

– [area bounded by $y = g(x)$, x-axis and the lines $x = a$, $x = b$]

where f(x) ≥ g(x) in [a , b]

Consider another case where $f(x) \geq g(x)$ in $[a, c]$ and $f(x) \leq g(x)$ in $[c, b]$, where $a < c < b$ (Fig) then the area of the regions bounded by the curves can be written as

Total Area = Area of the region ACBDA + Area of the region BPRQB

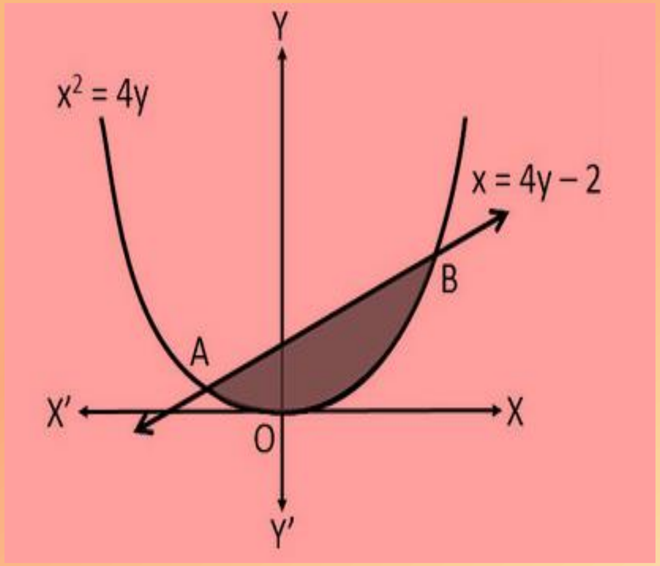


$$\text{Total Area} = \int_a^c [f(x) - g(x)]dx + \int_c^b [g(x) - f(x)]dx$$

EXAMPLES

Q1. Find the area bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$.

Solution:



We need to find the area of shaded region. First we find points A and B

$x^2 = 4y$…….(1)

$x = 4y - 2$ ……….(2)

From(1) and (2)

$$(4y-2)^2 = 4y \Rightarrow 16y^2 - 20y + 4 = 0$$

$$4y^2 - 5y + 1 = 0 \Rightarrow (4y-1)(y-1) = 0$$

$$\Rightarrow y = \frac{1}{4} \; or \; 1$$

Put values of y in eq (2), we get $x = -1 \; and \; 2$

Therefore the points are $A\left(-1, \frac{1}{4}\right) \; and \; B(2,1)$.

Required shaded area$=\int_{-1}^{2} y \, dx - \int_{-1}^{2} y \, dx$

$$=\int_{-1}^{2} \frac{x+2}{4} \, dx - \int_{-1}^{2} \frac{x^2}{4} \, dx$$

$$= \frac{1}{4}\int_{-1}^{2} [(x+2) - (x^2)] dx$$

$$=\frac{1}{4}\left[\left(\frac{x^2}{2} + 2x\right) - \left(\frac{x^3}{3}\right)\right]_{-1}^{2}$$

$$= \frac{1}{4}\left[\left(6 - \frac{8}{3}\right) - \left(-\frac{3}{2} + \frac{1}{3}\right)\right]$$

$$= \frac{1}{4}\left[\left(\frac{10}{3}\right) - \left(-\frac{7}{6}\right)\right] \quad \Rightarrow \frac{1}{4}\left[\frac{60+21}{18}\right] = \frac{1}{4} X \frac{81}{18}$$

$$= \frac{9}{8}\ sq\ units.$$

Q2.Find the area of the circle $4x^2 + 4y^2 = 9$ which is interior to the parabola $x^2 = 4y$

Solution:

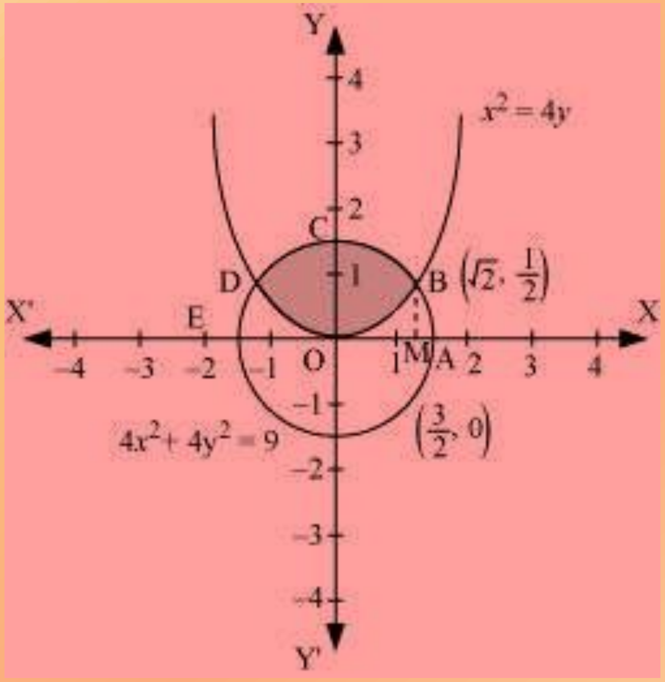
Solving the given equation of circle, $4x^2 + 4y^2 = 9$, and parabola, $x^2 = 4y$, we obtain the point of intersection as B $\left(\sqrt{2}\,,\frac{1}{2}\right)$and D ($figure$)

The required area is represented by the shaded area OBCDO. The area is symmetrical about y –axis

$\therefore$ Required area = 2 Area OBCO

Draw BM perpendicular to OA.

Therefore, the coordinates of M are $\left(\sqrt{2}, 0\right)$.



Area OBCO = Area OMBCO – Area OMBO

$$= \int_0^{\sqrt{2}} \sqrt{\frac{9-4x^2}{4}}\, dx - \int_0^{\sqrt{2}} \frac{x^2}{4}\, dx$$

$$= \frac{1}{2}\int_0^{\sqrt{2}} \sqrt{9-4x^2}\, dx - \frac{1}{4}\int_0^{\sqrt{2}} x^2\, dx$$

$$= \frac{1}{4}\left[x\sqrt{9-4x^2} + \frac{9}{2}sin^{-1}\left(\frac{2x}{3}\right)\right]_0^{\sqrt{2}} - \frac{1}{4}\left[\frac{x^3}{3}\right]_0^{\sqrt{2}}$$

$$= \frac{1}{4}\left[\sqrt{2} + \frac{9}{2}sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)\right] - \frac{1}{12}2\sqrt{2}$$

$$= \frac{\sqrt{2}}{4} + \frac{9}{8}sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) - \frac{\sqrt{2}}{6}$$

$$= \frac{\sqrt{2}}{12} + \frac{9}{8}sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$$

$$= \frac{1}{2}\left[\frac{\sqrt{2}}{6} + \frac{9}{4}sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)\right]$$

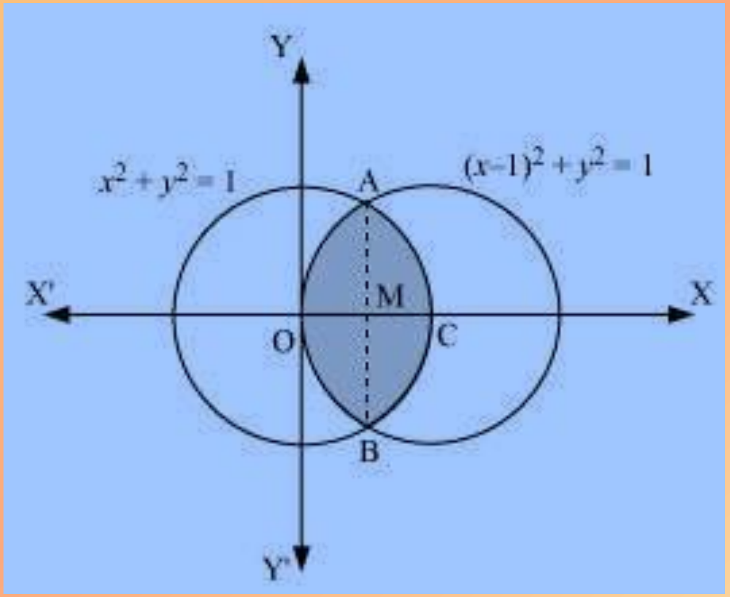
$\therefore$ Required area = 2 Area OBCO

$$= 2 \times \frac{1}{2}\left[\frac{\sqrt{2}}{6} + \frac{9}{4}sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)\right]$$

$$= \frac{\sqrt{2}}{6} + \frac{9}{4}sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) \text{ sq.units.}$$

Q3.Find the area bounded by curves $(x - 1)^2 + y^2 = 1$ and $x^2 + y^2 = 1$

Solution:



The area bounded by the curves, $(x - 1)^2 + y^2 = 1$ and $x^2 + y^2 = 1$, is represented by the shaded area. Solving the two equations we get the point of intersection as

A $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and B $\left(\frac{1}{2}, \frac{-\sqrt{3}}{2}\right)$.

The required area is symmetrical about x-axis.

∴ Area OBCAO = 2 × Area OCAO

Join AB, which intersects OC at M, such that AM is perpendicular to OC. The coordinates of M are $\left(\frac{1}{2}, 0\right)$

Area OCAO = Area OMAO +Area MCAM

$$= \int_0^{\frac{1}{2}} \sqrt{1-(x-1)^2}dx + \int_{\frac{1}{2}}^{1} \sqrt{1-x^2}\, dx$$

$$= \left[\frac{x-1}{2}\sqrt{1-(x-1)^2} + \frac{1}{2}sin^{-1}(x-1)\right]_0^{\frac{1}{2}} +$$

$$\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}sin^{-1}x\right]_{\frac{1}{2}}^{1}$$

$$= \left[\frac{-1}{4}\sqrt{1-\frac{1}{4}} + \frac{1}{2}sin^{-1}\left(\frac{-1}{2}\right) - \frac{1}{2}sin^{-1}(-1)\right] +$$

$$\left[\frac{1}{2}sin^{-1}(1) - \frac{1}{4}\sqrt{1-\frac{1}{4}} - \frac{1}{2}sin^{-1}\left(\frac{1}{2}\right)\right]$$

$$= \left[\frac{-\sqrt{3}}{8} - \frac{\pi}{12} + \frac{\pi}{4}\right] + \left[\frac{\pi}{4} - \frac{\sqrt{3}}{8} - \frac{\pi}{12}\right] = \frac{-\sqrt{3}}{4} - \frac{\pi}{6} + \frac{\pi}{2}$$

$$= \frac{\pi}{3} - \frac{\sqrt{3}}{4}$$

Therefore, required OBCAO = 2 × Area OCAO

$$= 2\left[\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right]$$

$$= \frac{2\pi}{3} - \frac{\sqrt{3}}{2} \text{ sq.units}$$

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THANK YOU

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